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Ising Model in a Quasiperiodic Transverse Field, Percolation, and Contact Processes in Quasiperiodic Environments

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Quantum Ising models in a transverse field are related to continuous-time percolation processes whose oriented percolation versions are contact processes. We study such models in the presence of quasiperiodic disorder and prove localization in the ground state, no percolation, and extinction, respectively, for sufficiently large disorder.

KEY WORDS: Quantum Ising model in quasiperiodic transverse field; percolation and contact processes in quasiperiodic environments; quasiperiodic disorder.

1. INTRODUCTION

Quantum Ising models in a transverse field are related by a Fortuin–Kasteleyn representation to continuous-time percolation processes whose oriented percolation version are contact processes.[1-3] These models have been studied in random environments[1-4]; we refer to ref. 2 for references from the physics literature. In this article we examine their behavior in the presence of quasiperiodic disorder (see the review in ref. 19) and prove localization in the ground state, no percolation, and extinction, respectively, for sufficiently large disorder.

We start by describing the models; we let $J > 0$ and $\mathbf{h} = \{h(x),\ x \in \mathbf{Z}^d\}$ with each $h(x) \geqslant 0$.

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1.1. Quantum Ising Model in a Transverse Field

The quantum spin Hamiltonian is

$$H = -\frac{J}{2} \sum_{\langle x, y \rangle} \sigma_3(x)\, \sigma_3(y) - \sum_x h(x)\, \sigma_1(x) \tag{1.1}$$

If $\Lambda \subset \mathbf{Z}^d$ is finite, we define H_Λ as the sum of terms in (1.1) indexed by sites and bonds within Λ. The finite-volume Hamiltonian has a unique ground state Ω_Λ and we can define the finite-volume correlation function

$$G_\Lambda^{(I)}((x, t), (y, s)) = \frac{(\Omega_\Lambda, \sigma_3(x)\, e^{-|t-s| H_\Lambda}\, \sigma_3(y) \Omega_\Lambda)}{(\Omega_\Lambda, e^{-|t-s| H_\Lambda} \Omega_\Lambda)}$$

for $x, y \in \mathbf{Z}^d$, t, $s \in \mathbf{R}$ (e.g., refs. 1 and 3). Since $G_\Lambda^{(I)}(\cdot, \cdot)$ is monotonically increasing in Λ, we can define

$$G^{(I)}(\cdot, \cdot) = \lim_{\Lambda \to \mathbf{Z}^d} G_\Lambda^{(I)}(\cdot, \cdot)$$

We will also write $G_{\mathbf{h}, J}^{(I)}(\cdot, \cdot)$ when we want to make explicit the dependence.

We will use $G^{(I)}(\cdot, \cdot)$ as an indicator of the amount of order or disorder in the system. When for some x we have that $G^{(I)}((x, 0), (y, 0))$ does not decay as $|y| \to \infty$, we say that the system exhibits long-range order (LRO) in the ground state. However, for an inhomogeneous system it will not in general be true that LRO is characterized by a uniform bound from below, but only that

$$\limsup_{|y| \to \infty} G^{(I)}((x, 0), (y, 0)) > 0$$

On the other hand, if for all x we have that $G^{(I)}((x, 0), (y, 0))$ decay as $|y| \to \infty$, we will say that the system exhibits localization in the ground state.

1.2. Continuous-Time Percolation Process

This percolation process is defined on $\mathbf{Z}^d \times \mathbf{R}$ as follows: Along each vertical line $\{x\} \times \mathbf{R}^d$ we put *cuts* at times given by a Poisson point process with intensity $h(x)$, and between each pair of adjacent vertical lines $\{x\} \times \mathbf{R}$ and $\{y\} \times \mathbf{R}$ (i.e., $\langle x, y \rangle$ is a bond) we place *bridges* at times given by a Poisson point process with intensity J. All these Poisson processes are independent of each other.

A configuration of the process is a realization of all these Poisson processes, i.e., a locally finite collection of *cuts* and *bridges*. We will denote by $\mathbf{Q} = \mathbf{Q}_{\mathbf{h},J}$ the percolation probability measure, i.e., the probability measure on the space of configurations.

Given a configuration of the process, we consider the subset of $\mathbf{Z}^{d+1}$ obtained by taking $\mathbf{Z}^d \times \mathbf{R}$, removing all *cuts* and adding all *bridges*, and decompose it into connected components which we call *clusters*. We say that $(x, t) \leftrightarrow (y, s)$ if they belong to the same cluster.

This inhomogeneous continuous-time percolation process appears implicitly in Campanino *et al.*[1] and was studied by Aizenman *et al.*[3] and by Klein.[4] The homogeneous version was considered by Bezuidenhout and Grimmett.[6]

We will denote by $C(x, t)$ the cluster to which (x, t) belongs; $|C(x, t)|$ will denote its measure on $\mathbf{Z}^d \times \mathbf{R}$, where $\mathbf{Z}^d$ is equipped with the counting measure and $\mathbf{R}$ with Lebesgue measure. We say that we have *percolation* if $\mathbf{Q}(|C(x, t)| = \infty) > 0$ for some (x, t) (and hence for all).

The connectivity function is defined by

$$G((x, t), (y, s)) = \mathbf{Q}((x, t) \leftrightarrow (y, s))$$

As in Section 1.1, we can talk about long-range order (LRO) or decay in the inhomogeneous system. Notice that

$$\mathbf{E}_{\mathbf{Q}}(|C(x, t)|) = \sum_{y \in \mathbf{Z}^d} \int ds\, G((x, t), (y, s)) \tag{1.2}$$

where $\mathbf{E}_{\mathbf{Q}}$ denotes expectation with respect to the probability measure $\mathbf{Q}$. It is well known that LRO implies percolation, and summable decay of the connectivity function [i.e., finiteness of the right-hand side of (1.2)] precludes percolation.

1.3. Contact Process

If we consider the oriented percolation process we obtain by keeping the *cuts* as above, but replacing the *bridges* by *one-way bridges*, i.e., each Poisson process of bridges between pairs of adjacent vertices lines $\{x\} \times \mathbf{R}$ and $\{y\} \times \mathbf{R}$ is replaced by two independent Poisson processes with the same intensity J, the first giving one-way bridges from $\{x\} \times \mathbf{R}$ to $\{y\} \times \mathbf{R}$, and the second from $\{y\} \times \mathbf{R}$ to $\{x\} \times \mathbf{R}$, and uncut segments can only be traversed in the direction of increasing time, we obtain the graphical representation of the inhomogeneous contact process.[6]

In the contact process language, $(x, t) \to (y, s)$ means that (x, t) infects (y, s), i.e., there is a path from (x, t) to (y, s) made up of uncut segments of vertical lines, traversed in the direction of increasing time, and one-way bridges. Let

$$D(x, t) = \{(y, s); (x, t) \to (y, s)\}$$

be the *infected cluster* of (x, t), and let $D(x, t; s) = D(x, t) \cap (\mathbf{Z}^d \times \{s\})$. We say that we have *survival* of the infection if $\mathbf{Q}\{D(x, t; s) \neq \emptyset$ for all $s \geqslant t\} > 0$ for some (and hence for all) (x, t), otherwise we have *extinction.*

Clearly, survival of the contact process (with parameters $\mathbf{h}, J$) can only happen if we have percolation (with parameters $\mathbf{h}, 2J$).

The contact process in a random environment has been studied by Liggett,[7,8] Bramson *et al.*,[9] Andjel,[10] and Klein.[4]

It follows from the Fortuin–Kasteleyn representation of classical Ising models (e.g., ref. 11) and the results in refs. 1 and 3 that

$$G_{2\mathbf{h}, J/2}((x, t), (y, s)) \leqslant G^{(\mathrm{I})}_{\mathbf{h}, J}((x, t), (y, s)) \leqslant G_{\mathbf{h}, J}((x, t), (y, s))$$

We will thus study the continuous-time percolation model.

We introduce *quasiperiodic disorder* by taking

$$h(x) = f(Ax + \theta)$$

for all $x \in \mathbf{Z}^d$, where $\theta \in \mathbf{T}^k$, the k-dimensional torus, $f: \mathbf{T}^k \to [0, \infty)$, and A is a $k \times d$ real matrix such that $x \to T^x$, defined by $T^x\theta = Ax + \theta$, gives an ergodic action of $\mathbf{Z}^d$ on $\mathbf{T}^k$.

If f is bounded from above [e.g., if $f \in C(\mathbf{T}^k)$], we always have LRO for large J by comparison with the homogeneous case.[1,2] If f is bounded away from zero [i.e., $f(\theta) \geqslant \delta > 0$ for some δ], we always have exponential decay of $G((x, t), (y, s))$ for small J for the same reason.

But if f can take arbitrarily small values, there will be (for a.e θ) infinitely many regions in which the system wants to be ordered, as in the phenomenon of Griffiths singularities, even for arbitrarily small J. This is the situation we study in this article.

We will need some definitions; we always take $\eta > 0$.

Definition. We say that $g \in C(\mathbf{T}^k)$ is of type η if $g(\theta) \geqslant 0$ and $g^{-1}(\{0\})$ is a finite set $\{\theta_1,..., \theta_R\}$ with

$$\liminf_{\theta \to \theta_i} e^{|\theta - \theta_i|^{-\eta}} g(\theta) > 0$$

for $i = 1,..., R$.

Typical examples are nonnegative analytic funuctions, e.g.,

$$g(\theta) = \prod_{j=1}^{k} [1 - \cos 2\pi(\xi_j \theta_j)] \qquad \text{with} \quad \xi \in \mathbf{R}^k$$

which are of type η for all $\eta > 0$.

Definition. $f\colon \mathbf{T}^k \to [0, \infty)$ is η-admissible if there exists g of type η such that $f(\theta) \geqslant g(\theta)$ for all $\theta \in \mathbf{T}^k$.

Definition. We say that a real $k \times d$ matrix A is diophantine (or has typical diophantine properties) if there exists $\varepsilon > 0$ and $C > 0$ such that

$$d(Ax + \theta, \theta) \geqslant \frac{C}{|x|^{d+\varepsilon}} \tag{1.3}$$

for all $\theta \in \mathbf{T}^k$, $x \in \mathbf{Z}^d \backslash \{0\}$, where $d(\cdot, \cdot)$ denotes the distance in $\mathbf{T}^k$. By "A is ε-diophantine" we mean that (1.3) holds for the specified ε with some $C = C_{A,\varepsilon} > 0$.

Our first theorem is:

Theorem 1.1. Let $d = 1, 2,..., k = 1, 2,....$ Let $h(x) = f(Ax + \theta)$, where A is ε-diophantine and f is η-admissible, with $0 < \eta < 1/(d + \varepsilon)$. Then for any $m > 0$ and any v, with $(d + \varepsilon)\eta < v < 1$, there exists $J_1 = J_1(d, k, \varepsilon, C_{A,\varepsilon}, \eta, m, v) > 0$ such that, if $0 < J < J_1$, we have that for almost every $\theta \in \mathbf{T}^k$ and all $x \in \mathbf{Z}^d$,

$$G((x, t), (y, s)) \leqslant C_{x,\theta} \exp(-m\{|x - y| + [\log(1 + |t - s|)]^{1/v}\})$$

for all $y \in \mathbf{Z}^d$, $t, s \in \mathbf{R}$, with $C_{x,\theta} < \infty$. In particular we have extinction of the contact process for almost every θ if J is sufficiently small.

If $k = d = 1$, the matrix A can be identified with a real number, say w, and in this case $T^x\theta = wx + \theta$. The ergodicity condition is equivalent to requiring w to be irrational. The diophantine condition is the usual one for real numbers.

There is an analogy between localization in the ground state of quantum spin systems with disorder and localization for disordered Schrödinger operators (e.g., refs. 12 and 1). It is easier to prove localization for random Schrödinger operators (e.g., ref. 13) than localization in the ground state of an Ising model with a random transverse field.[1,4] But for quasiperiodic disorder, the proof of Theorem 1.1 is not only easier than the proof of localization for quasiperiodic Schrödinger operators,[14–16] but there is no difference between one or many frequencies. In fact, many frequencies make localization in the ground state of an Ising model with a quasiperiodic transverse field more likely.

But the analogy can only be taken so far. It is known that for the one-dimensional almost-Mathieu operator $H = -J\Delta + \cos 2\pi(wx + \theta)$ one gets very different behavior for diophantine w or Liouville w (e.g., ref. 17). For any irrational w the Lyapunov coefficient is always positive for $J < 1/2$, but while we have localization for w diophantine (at least for $J \ll 1$), if w is a Liouville number the spectrum is always singular continuous for $J < 1/2$. But in our case, it turns out that for $k = d = 1$ we have localization in the ground state of the Ising model with a quasiperiodic transverse field for any irrational w, at least for η-admissible functions with $\eta < 1/3$. We do, however, loose the faster than polynomial decay in the time direction of Theorem 1.1, and the proof is much harder.

Theorem 1.2. Let $d = 1$, $0 < \eta < 1/3$. Let $h(x) = f(wx + \theta)$, where w is an arbitrary irrational number and f is η-admissible. Then for any $m > 0$ there exists $J_2 = J_2(m, \eta, w) > 0$ such that, if $0 < J < J_2$, the conclusions of Theorem 1.1 hold with $\nu = 1$.

2. THE MULTISCALE ANALYSIS

We will use the same scheme for the multiscale analysis as in refs. 5, 1, and 4. Let us consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ in an inhomogeneous environment. We let

$$\Lambda_L(x) = \{ y \in \mathbf{Z}^d;\ |y - x|_\infty < L \}$$

and

$$B_L(x, t) = \Lambda_L(x) \times [t - e^{T(L)}, t + e^{T(L)}]$$

where $T: \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function to be specified later. We also let

$$\partial \Lambda_L(x) = \{ y \in \Lambda_L(x);\ \langle y, y' \rangle \subset \mathbf{Z}^d \text{ for some } y' \notin \Lambda_L(x) \}$$

$$\partial_H B_L(x, t) = \Lambda_L(x) \times \{ t - e^{T(L)}, t + e^{T(L)} \}$$

$$\partial_V B_L(x, t) = \partial \Lambda_L(x) \times [t - e^{T(L)}, t + e^{T(L)}]$$

$$\partial B_L(x, t) = \partial_H B_L(x, t) \cup \partial_V B_L(x, t)$$

Definition. Let $m > 0$, $L > 1$. A site $x \in \mathbf{Z}^d$ is called (m, L)-regular if

$$G_{B_L(x,0)}((x, 0), Y) \leqslant e^{-mL}$$

for all $Y \in \partial B_L(x, 0)$. Otherwise x is called (m, L)-singular.

Due to the translational invariance in the t direction, we might have taken in this definition every box of the form $B_L(x, t)$ for any t as well.

Definition. A set $A \subset \mathbf{Z}^d$ is called (m, L)-regular if every $x \in A$ is (m, L)-regular. Otherwise it is called (m, L)-singular.

Let us fix μ, with $0 < \mu < 1$.

Definition. A site $x \in \mathbf{Z}^d$ is called L-resonant if $h(x) < e^{-L^\mu}$. A set $A \subset \mathbf{Z}^d$ is called L-resonant if there exists $x \in A$ which is L-resonant.

Definition. A number $L > 0$ will be called m-simple if for any $x \in \mathbf{Z}^d$ which is (m, L)-singular we must have that $\Lambda_L(x)$ is L-resonant.

We will use two "standard" multiscale analysis statements (e.g., ref. 4), which we need to formulate in a slightly more general form. We will consider stationary disordered environments, i.e., $\{h(x),\ x \in \mathbf{Z}^d\}$ is a stationary stochastic process. In the case of quasiperiodic environments, we have $\mathbf{T}^k$ with normalized Lebesgue measure as our underlying probability space.

Theorem 2.1. Consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ in a stationary disordered environment. Let $T(L) = e^{L^\nu}$ with $0 < \nu < 1$, take $m_\infty > 0$, and set

$$P_L = \mathbf{P}\{0 \text{ is } (m_\infty, L)\text{-regular}\}$$

Suppose there exists an increasing sequence of scales L_k, with

$$\frac{L_{k+1}}{L_k} < e^{L_k^\nu}$$

such that

$$\sum_{k=1}^{\infty} L_{k+1}^d (1 - P_{L_k}) < \infty$$

Then for any m with $0 < m < m_\infty$ we have, with probability one, that for every $x \in \mathbf{Z}^d$,

$$G((x, t), (y, s)) \leqslant C_x(\mathbf{h}, m) \exp(-m\{|x - y| + [\log(1 + |t - s|)]^{1/\nu}\})$$

for all $y \in \mathbf{Z}^d$, $t, s \in \mathbf{R}$, with $C_x(\mathbf{h}, m) < \infty$.

Theorem 2.2. Consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ in an inhomogeneous environment. Let $L_0 \leqslant L_1 < (1/2R) L_2$ and suppose there exists $x_1, ..., x_R \in \Lambda_{L_2}(y)$ such that $\Lambda_{L_2}(y) \setminus \bigcup_{i=1}^R \Lambda_{L_1}(x_i)$ is (m, L_0)-regular. Then

$$G_{B_{L_2}(y,0)}(y, Y) \leqslant \exp(-\bar{m} L_2)$$

for all $Y \in \partial_V B_L(y, 0)$ with

$$\bar{m} = \bar{m}(m, L_0, L_1, L_2) = \left(m - \frac{(d-1)\ln L_0}{L_0} - \frac{\ln T(L_0)}{L_0}\right)\left(1 - \frac{L_1 R + 1}{L_2}\right) \tag{2.1}$$

The statement of Theorem 2.2 makes sense only if the right-hand side of (2.1) remains positive, which will be always true for large enough L_0 for suitable choices of $T(L)$.

The estimation of $G_{B_L(y,0)}(y, Y)$ for $Y \in \partial_H B_L(y, 0)$ is as always much more complicated. In some cases we will find the following theorem particularly useful.

Theorem 2.3. Consider the continuous-time percolation process on $\mathbf{Z}^d \times \mathbf{R}$ is an inhomogeneous environment. Let $L_0 = L_1$, $L_2 = L_0^\gamma$ with $\gamma > 1$, $T(L) = e^{L^\nu}$, $0 < \mu < \nu < 1$. Suppose L_0 is large enough and:

(a) The event described in Theorem 2.2 occurs.

(b) $\Lambda_{L_2}(y)$ is L_2-nonresonant.

Then $\Lambda_{L_2}(y)$ is $(\bar{m}, L_2)$-regular with

$$\bar{m} = m\left(1 - \frac{L_0 R + 1}{L_0^\gamma}\right) - \frac{2}{L_0^{\gamma - 1}} - \frac{2}{L_0^{1-\nu}}$$

Theorems 2.1–2.3 can be proved by repeating the proofs of the analogous theorems in refs. 1, 4, and 5. They are the key technical steps in the multiscale analysis.

3. PROOF OF THEOREM 1.1

The following proposition contains the property of an η-admissible function that is actually used in the proofs.

Proposition 3.1. Let f be an η-admissible function. Then there exist $\theta_1, \ldots, \theta_R \in \mathbf{T}^k$ such that, for any $\mu > 0$, if L is sufficiently large, we have that $f(\theta) < e^{-L^\mu}$ implies $d(\theta, \theta_i) < 2L^{-\mu/\eta}$ for some $i = 1, \ldots, R$.

We now restrict ourselves to a continuous-time percolation process in a quasiperiodic environment with $h(x) = f(Ax + \theta)$, A and f as in Theorem 1.1. We fix μ such that $(d + \varepsilon)\eta < \mu < 1$, and $m > 0$. We set $p = \mu/\eta$, and notice that $p > d + \varepsilon$. We take $1 < \gamma < p/(d + \varepsilon)$.

Lemma 3.2. Suppose $L > 0$ is m-simple and sufficiently large. Then for any $x \in \mathbf{Z}^d$ there exist $y_1, \ldots, y_R \in \Lambda_{L^\gamma}(x)$ such that $\Lambda_{L^\gamma}(x) \setminus \bigcup_{i=1}^R \Lambda_L(y_i)$ is (m, L)-regular.

Proof. Suppose $x_1, x_2,..., x_{R+1} \in \mathbf{Z}^d$ are (m, L)-singular. Since L is m-simple, that means that $\Lambda_L(x_i)$ is L-resonant, $i = 1,..., R+1$, and due to Proposition 3.1 there exist $\bar{x}_i \in \Lambda_L(x_i)$ and $j_i \in \{1,..., R\}$, $i = 1,..., R+1$, such that

$$d(T^{\bar{x}_i}\theta, \theta_{j_i}) < 2L^{-p}$$

Thus there exist $\bar{x}_{k_1}, \bar{x}_{k_2}$ such that

$$d(T^{\bar{x}_{k_1}}\theta, T^{\bar{x}_{k_2}}\theta) < 4L^{-p}$$

But according to (1.3),

$$d(T^{\bar{x}_{k_1}}\theta, T^{\bar{x}_{k_2}}\theta) > \frac{C}{|\bar{x}_{k_1} - \bar{x}_{k_2}|^{d+\varepsilon}}$$

It follows that

$$|\bar{x}_{k_1} - \bar{x}_{k_2}| > \left(\frac{C}{4}\right)^{1/(d+\varepsilon)} L^{p/(d+\varepsilon)} > 2L^{\gamma} + 2L$$

if L is large enough, since $\gamma(d+\varepsilon) < p$. Thus $|x_{k_1} - x_{k_2}| > 2L^{\gamma}$, which proves the lemma. ∎

Lemma 3.3. Let L be m-simple. Then L^{γ} is $\bar{m}$-simple with

$$\bar{m} = m\left(1 - \frac{2R}{L^{\gamma-1}}\right) - \frac{2}{L^{\gamma-1}} - \frac{2}{L^{1-\nu}}$$

for all L sufficiently large.

Proof. It follows from Theorem 2.3 and Lemma 3.2. ∎

We can now prove Theorem 1.1. Let us pick an initial scale L_1, sufficiently large so we can apply the previous lemmas, and let $L_{i+1} = L_i^{\gamma}$ for $i = 1, 2,...$. Let $m_1 > 0$; we set

$$m_{i+1} = m_i\left(1 - \frac{2R}{L_i^{\gamma-1}}\right) - \frac{2}{L_i^{\gamma-1}} - \frac{2}{L_i^{1-\nu}}$$

Given $m_\infty > 0$, we can find $m_1 > 0$ such that $m_i > m_\infty$ for all $i = 1, 2,...$. If we take J sufficiently small, we can guarantee that L_1 is m_1-simple. It follows from Lemma 3.3 that L_i is m_∞-simple for any $i = 1, 2,...$. Thus

$$\mathbf{P}\{0 \text{ is } (m_\infty, L_i)\text{-singular}\} \leqslant \mathbf{P}\{\Lambda_{L_i}(0) \text{ is } L_i\text{-resonant}\} \leqslant \frac{4R}{L_i^p}$$

by Proposition 3.1.

Theorem 1.1 now follows from Theorem 2.1. ∎

4. PROOF OF THEOREM 1.2

For this proof we need to make certain changes in the multiscale scheme. To simplify the argument we will take $R=1$, $\theta_1=0$; the proof extends to the general case with the obvious modifications.

Let $w=[k_1, k_2,...]$ be the continuous-fraction expansion of w; we call $p_n/q_n=[k_1, k_2,..., k_n]$ the nth approximant. We will denote $|q_n w - p_n| = d(T^{q_n}0, 0)$ by Δ_n. We are going to use the following properties of continuous-fractions expansion (see, e.g., ref. 18):

1. We have

$$\Delta_n \geqslant \frac{1}{q_{n+1}}\left(1-\frac{q_n}{q_{n+2}}\right) \geqslant \frac{1}{2q_{n+1}} \tag{4.1}$$

2. For every $0<l<q_{n+1}$, $\theta\in\mathbf{T}^1$,

$$d(\theta, T^l\theta)\geqslant \Delta_n \tag{4.2}$$

3. We have

$$q_n \geqslant (\sqrt{2})^{n-1} \tag{4.3}$$

Let us pick an initial scale L_1. We set our sequence of scales L_i, $i=1, 2,...$, by the following inductive rule: Fix

$$s>1, \qquad 1<\gamma<s, \qquad r<\min\left(\frac{s-\gamma}{2}, s(\gamma-1), \frac{s-1}{3}\right)$$

Given i, find $n(i)$ such that

$$q_{n(i)} \leqslant L_i < q_{n(i)+1}$$

If $q_{n(i)+1} < q^s_{n(i)}$ put $L_{i+1}=L_i^\gamma$, otherwise put $L_{i+1}=L_i q^r_{n(i)}$. Using our condition on r, we get that if L_1 is sufficiently large, then for each j such that $q_{j+1}>q_j^s$ there exists at least one $i(j)$ such that

$$q_j < L_{i(j)} < L_{i(j)+1} < L_{i(j)+2} < q_{j+1}/2$$

We also need to change our definition of $T(L)$. If $L_{i+1}=L_i^\gamma$, we define as before $T(L_{i+1})=\exp(L^\nu_{i+1})$. If $L_{i+1}=L_i q^r_{n(i)}$, we put $T(L_{i+1})=\exp(L_{i+1} q^{-r\delta}_{n(i)})$ for some δ, with $1>\delta>0$, to be specified.

Let

$$j_0(n)=\max\left\{j: L_j<\frac{q_n}{2}\right\}$$

$$j_1(n)=\max\{j: L_j<q_n\}$$

Lemma 4.1. We can choose μ, ν, s, γ, r in such a way that for any $m>0$ there exists J_0 such that, if $J<J_0$, we have that L_i is m-simple if $i=j_1(n)$ for some n.

Lemma 4.1 is a special case of Lemma 4.3, which is proved in Section 5.

Recall $p=\mu/\eta$. We will always have $p>\max(\gamma, r+1)$.

Lemma 4.2. For a.e. $\theta\in\mathbf{T}$ and all $x\in\mathbf{Z}^d$, there exists $k(\theta, x)<\infty$ such that for $k>k(\theta, x)$ we have that $\Lambda_{L_{k+1}}(x)$ is (L_k, m)-regular if L_k is m-simple.

Proof. If L_k is m-simple, it follows from Proposition 3.1 that

$$\begin{aligned} p_k &\equiv \mathbf{P}\{\Lambda_{L_{k+1}}(x) \text{ is } (L_k, m)\text{-singular}\} \\ &\leqslant \mathbf{P}\{\Lambda_{L_{k+1}+2L_k}(x) \text{ is } L_k\text{-resonant}\} \\ &\leqslant 2(L_{k+1}+2L_k)\frac{4}{L_k^p} \leqslant \frac{16L_{k+1}}{L_k^p} \end{aligned}$$

The lemma now follows from the Borel–Cantelli lemma, since $\sum_{k=1}^{\infty}(L_{k+1}/L_k^p)<\infty$ as $p>\gamma$, $p>r+1$. ∎

Lemmas 4.1 and 4.2 already give decay for $G((x, t), (y, s))$, but with no information on the rate of decay. Indeed, it follows that we have an increasing sequence of scales L_{k_i} such that for $L_{k_i}<|y-x|<L_{k_{i+1}}$ we have

$$|G((x, t), (y, s))| \leqslant e^{-(m/2)L_{k_i}}$$

for k_i sufficiently large.

To obtain the decay of Theorem 1.2 we need to control—in both deterministic and probabilistic ways—the growth of the sequence L_{k_i}. We will actually need the following more detailed version of the Lemma 4.1, whose proof we will postpone to the next section.

Lemma 4.3. Suppose $0<\eta<1/3$, and let μ and δ be such that

$$p>\max\left\{\gamma s, \frac{s^2}{s-r}, \gamma(3r+1), \frac{(3r+1)s}{s-r}\right\}$$

$$0<\delta<\min\left\{\frac{s(1-\mu)}{2r}, \frac{1-\eta}{r\eta}, \frac{1-3\eta}{2-\eta}, \frac{1}{2}\right\}$$

Then for any $m>0$ there exists J_0 such that for $J<J_0$ the scale L_i is m-simple in the following cases:

1. If $i = j_1(n)$ for some n.
2. If $q_{n(i-1)+1} < q^s_{n(i-1)}$ and $n(i-1) = n(i)$.
3. If $q_{n(i-1)+1} < q^s_{n(i-1)}$, $n(i-1) < n(i)$, and $q_{n(i)+1} < q^s_{n(i)}$.
4. If $i = j_0(n) - 1$ or $j_0(n)$ for some n such that $q_{n(i)} > q^s_{n(i)-1}$.
5. If $q_{n(i-1)+1} > q^s_{n(i-1)}$ and $q_{n(i)+1} < q^s_{n(i)}$.

We will also need the following two lemmas.

Lemma 4.4. For every $x \in \mathbf{Z}$ and $k > 1$ the set of phases

$$\Theta^k_x = \left\{ \theta \in \mathbf{T}^1\text{: there exists } y \in \mathbf{Z},\ L_k\text{-resonant, such that } |y - x| < \frac{q_{n(k)+1}}{n(k)^2} \right\}$$

has measure not exceeding $2/n(k)^2$.

Proof. Let us consider the set

$$B^k_x = \left\{ \theta\text{: for all } y \in [x, x + q_{n(k)}] \text{ we have } d(T^y\theta, 0) > \frac{1}{q_{n(k)} n(k)^2} \right\}$$

We will prove that for $\theta \in B^k_x$, $d(T^y\theta, 0) > 1/q_{n(k)} n(k)^2$ for any $x - q_{n(k)+1}/n(k)^2 \leqslant y \leqslant x + q_{n(k)+1}/n(k)^2$. Indeed, let $l = [(y-x)/q_{n(k)}]$. Then by (4.1) and (4.2)

$$d(T^y\theta, T^{y - lq_{n(k)}}\theta) = |l\Delta_{n(k)}| \leqslant \frac{|l|}{2q_{n(k)+1}}$$

On the other hand, we have $y - lq_{n(k)} \in [x, x + q_{n(k)}]$. Thus $d(T^{y - lq_{n(k)}}\theta, 0) > 1/q_{n(k)} n(k)^2$. For $|y - x| \leqslant q_{n(k)+1}/n(k)^2$ we have $|l| \leqslant q_{n(k)+1}/q_{n(k)} n(k)^2$ and $d(T^y\theta, 0) > 1/2q_{n(k)} n(k)^2 > 1/L^p_k$. That proves the inclusion $B^k_x \subset (\Theta^k_x)^c$. Evidently

$$\mathbf{P}((B^k_x)^c) \leqslant \sum_{y \in [x, x + q_{n(k)}]} \mathbf{P}\left\{\theta\text{: } d(T^y\theta, 0) < \frac{1}{q_{n(k)} n(k)^2}\right\} \leqslant \frac{2}{n(k)^2} \quad \blacksquare$$

Lemma 4.5. Let $d_k = q_{n(k)+1}/n(k)^2 - 2L_k$. Under the same conditions as in Lemma 4.1 we have that for a.e. θ and every $x \in \mathbf{Z}$ there exists $k_0(x, \theta) < \infty$ such that for $k > k_0(x, \theta)$, we have that $\Lambda_{d_k}(x)$ is (m, L_k)-regular.

Proof. Suppose $n(k_1) = n(k_2)$. Then using the notations of the proof of Lemma 4.4 we have by definition $B^{k_1}_x = B^{k_2}_x$. It follows that $\bigcup_{k_1 : n(k_1) = n(k)} \Theta^{k_1}_x \subset (B^k_x)^c$.

Let $\bar{k} = \min\{k_1 : n(k_1) = n(k)\}$. Let y be an (m, L_k)-singular point. Then if k is sufficiently large we conclude that $\Lambda_{L_k}(y)$ contains at least one $(2m, L_{\bar{k}-1})$-singular point y_0. But it follows from Lemma 4.1 that for J small enough the scale $L_{\bar{k}-1}$ is $2m$-simple. Thus there exists an $L_{\bar{k}-1}$-resonant point

$$y_1 \in \Lambda_{L_{k-1}}(y_0)$$

For $\theta \in B_x^k$ we have $d(T^z\theta, 0) > 1/2q_{n(k)}n(k)^2$ for $|z-x| \leqslant [q_n(k)+1]/n(k)^2$. We have two cases:

(i) Suppose $L_{\bar{k}} = L_{\bar{k}-1} q^r_{n(\bar{k}-1)}$. Then we have $L_{\bar{k}} > q_{n(k)} > q^s_{n(\bar{k}-1)}$ and $1/2q_{n(k)}n(k)^2 > 1/L^p_{\bar{k}-1}$ if $p > s/(s-r)$.

(ii) Now suppose $L_{\bar{k}} = L^\gamma_{\bar{k}-1}$. Then

$$\frac{1}{2q_{n(k)}n(k)^2} > \frac{1}{2L_{\bar{k}}n(k)^2} = \frac{1}{2L^\gamma_{\bar{k}-1}n(k)^2} > \frac{1}{L^p_{\bar{k}-1}}$$

if $p > \gamma$.

We conclude that for $\theta \in B_x^k$ we have $d(x, y_1) > q_{n(k)+1}/n(k)^2$, which implies

$$d(x, y) > \frac{q_{n(k)+1}}{n(k)^2} - L_{\bar{k}} - L_k > \frac{q_{n(k)+1}}{n(k)^2} - 2L_k$$

It now suffices to use the Borel–Cantelli lemma and Lemma 4.4 to get the statement of Lemma 4.5. ∎

We can now prove Theorem 1.2 assuming Lemma 4.3. Fix $x \in \mathbf{Z}$, $b > 1$. Let $|y-x|$ be large enough. Suppose $(y, t_1) \in B_{bL_{k+1}}(x, t) \backslash B_{bL_k}(x, t)$. We have two cases:

1. $bL_{k+1} < q_{n(k)+1}/n(k)^2 - 2L_k$.
2. $bL_{k+1} \geqslant q_{n(k)+1}/n(k)^2 - 2L_k$.

Suppose θ belongs to the set of full measure $\bigcup_{k'=1}^\infty \bigcap_{k>k'} B_x^k$ and $k > k_0(x, \theta)$. Then in the first case $\Lambda_{bL_{k+1}}(x)$ is an (m, L_k)-regular region and applying, say, the proof of Theorem 3.3 in ref. 4, we get the desired decay of the two-point function. In the second case we have

$$L_{k+2} \geqslant L_{k+1} q^r_{n(k)} > n(k)^2(bL_{k+1} + 2L_k) \geqslant q_{n(k)+1}$$

We have four subcases:

1. $L_{k+1} > q_{n(k)+1}$.
2. $L_{k+1} < q_{n(k)+1}$, $q_{n(k)+1} > q^s_{n(k)}$.

3. $L_{k+1} < q_{n(k)+1}$, $q_{n(k)+1} < q^s_{n(k)}$, $L_k = L^{\gamma}_{k-1}$ (notice we must also have $L_{k+1} = L^{\gamma}_k$).

4. $L_{k+1} < q_{n(k)+1}$, $q_{n(k)+1} < q^s_{n(k)}$, $L_k = L_{k-1} q^r_{n(k-1)}$.

In each of these subcases we may apply Lemma 4.3 to get that L_k is m-simple. Now suppose $k > \max(k_0(x, \theta), k(x, \theta))$ [see Lemmas 4.2 and 4.5]. The same argument as before applies to prove exponential decay. ∎

We required some conditions on p in Lemmas 4.2–4.5. These can be satisfied if

$$p > \max\left\{\gamma(3r+1), \frac{s(3r+1)}{s-r}, \frac{s^2}{s-r}, \gamma\right\}$$

Given $\varepsilon > 0$, we may pick

$$s > 1, \qquad 1 < \gamma < s, \qquad r < \min\left(\frac{s-\gamma}{2}, s(\gamma-1)\right)$$

in such a way that we can choose $p < 1 + \varepsilon$. That means that for any $0 < \eta < 1$ we can find μ such that Lemmas 4.2–4.5 hold with $p = \mu/\eta$. Thus the only restriction on η follows from the conditions on δ in Lemma 4.3 and it is $\eta < 1/3$.

5. PROOF OF LEMMA 4.3

Given m, L_0, L_1, L_2, we define $\bar{m}(m, L_0, L_1, L_2)$ by (2.1). We take $p > \gamma s$.

Lemma 5.1. Suppose L_i is m_i-simple. If either one of:

(i) $q_{n(i)+1} < q^s_{n(i)}$ and $n(i) = n(i+1)$

(ii) $q_{n(i)+1} < q^s_{n(i)}$, $n(i) < n(i+1)$, and $q_{n(i+1)+1} < q^s_{n(i+1)}$

(iii) $q_{n(i)+1} > q^s_{n(i)}$, $q_{n(i+1)+1} < q^s_{n(i+1)}$

holds, then L_{i+1} is m_{i+1}-simple with $m_{i+1} = \bar{m}(m_i, L_i, L_i, L_{i+1})$. If:

(iv) $q_{n(i+1)+1} > q^s_{n(i+1)}$

then $L_{j_0(n(i+1))-1}$ is $m_{j_0(n(i+1))-1}$-simple and $L_{j_0(n(i+1))}$ is $m_{j_0(n(i+1))}$-simple with

$$m_{j_0(n(i+1))-1} = \bar{m}(m_i, L_i, L_{j_0(n(i+1))-2}, L_{j_0(n(i+1))-1})$$
$$m_{j_0(n(i+1))} = \bar{m}(m_{j_0(n(i+1))-1}, L_i, L_{j_0(n(i+1))}, L_{j_0(n(i+1))})$$

We now proceed to finish the proof of Lemma 4.3 assuming Lemma 5.1. We take

$$p > \max\left(\frac{s}{s-2r}, \frac{\gamma s}{s-r\gamma}\right)$$

Lemma 5.2. For any n the scale $L_{j_1(n)}$ is m_{j_1}-simple with $m_{j_1} = \bar{m}(m_{j_1-1}, L_{j_1-1}, L_{j_1-1}, L_{j_1})$.

Proof. We will prove the lemma by induction in n. Suppose $L_{j_1(n-1)}$ is $m_{j_1(n-1)}$-simple.

1. If $j_1(n) = j_0(n)$, $q_n > q^s_{n-1}$, then taking $i = j_1(n-1)$, we can apply Lemma 5.1, case (iv).

2. If $j_1(n) > j_0(n)$, $q_n > q^s_{n-1}$, then $L_{j_1} = L_{j_0} q^r_{n-1} < q_n < L_{j_1} q^r_{n-1}$, and by (iv) of Lemma 5.1 the scale L_{j_0} is m_{j_0}-simple. For two points x_1, x_2 that are L_{j_0}-resonant we have by Proposition 3.1

$$d(T^{x_1}\theta, T^{x_2}\theta) < \frac{4}{L^p_{j_0}} \leqslant \frac{4q^{rp}_{n-1}}{L^p_{j_1}} < \frac{4q^{2rp}_{n-1}}{q^p_n} < \frac{1}{q_n^{p(1-2r/s)}}$$

On the other hand, $d(T^{x_1}\theta, T^{x_2}\theta) > 1/2q_n$ for $|x_1 - x_2| < q_n$ by (4.2). Since $p > s/(s-2r)$, we can conclude that $|x_1 - x_2| \geqslant q_n$.

Suppose there exist three (L_{j_0}, m_{j_0})-singular points $x_1, x_2, x_3 \in \Lambda_{L_{j_1}}$. Since L_{j_0} is m_{j_0}-simple, we can find L_{j_0}-resonant points $\bar{x}_1, \bar{x}_2, \bar{x}_3$, with $\bar{x}_i \in \Lambda_{L_{j_0}}(x_i)$, $i = 1, 2, 3$. Thus $|\bar{x}_i - \bar{x}_j| \geqslant q_n$, $1 \leqslant i < j \leqslant 3$. But at least for one pair (i, j) the distance $|\bar{x}_i - \bar{x}_j| < L_{j_1}/2 + L_{j_0} + 1 < L_{j_1} < q_n$.

This contradiction proves that assumptions of the Theorem 2.3 are satisfied; thus, assuming that $\Lambda_{L_{j_1}}$ is nonresonant, we apply Theorem 2.3 to prove that it is (L_{j_1}, m_{j_1})-regular, which proves that the scale L_{j_1} is m_{j_1}-simple.

3. If $q_n < q^s_{n-1}$ and $L_{j_1} = L_{j_1-1} q^r_{n(j_1-1)}$, then $L^\gamma_{j_1} > q_n$ and $L_{j_1-1} = L_{j_1(n-1)}$; therefore it is $m_{j_1(n-1)}$-simple.

For L_{j_1-1}-resonant points x_1, x_2 we can now use

$$d(T^{x_1}\theta, T^{x_2}\theta) < \frac{4}{L^p_{j_1-1}} < \frac{4q^{rp}_{n(j_1-1)}}{L^p_{j_1}} < q_n^{-p(1/\gamma - r/s)}$$

Since $p > \gamma s/(s - r\gamma)$, we can now use the same argument as above.

4. The last case is $q_n < q^s_{n-1}$ and $L_{j_1} = L^\gamma_{j_1-1}$. If $n(j_1-1) = n(j_1)$, we can apply cases (i)–(iii) of Lemma 5.1. If $n(j_1-1) < n(j_1)$, then $j_1 - 1 =$

$j_1(n(j_1-1)+1)$; thus L_{j_1-1} is m_{j1-1}-simple. For L_{j_1-1}-resonant points x_1, x_2 we get

$$d(T^{x_1}\theta, T^{x_2}\theta) < \frac{4}{L^p_{j_1-1}} = \frac{4}{L^{p/\gamma}_{j_1}} < \frac{4}{q_n^{p/\gamma^2}}$$

$$d(T^{x_1}\theta, T^{x_2}\theta) > \frac{1}{2q_n} \qquad \text{if} \quad |x_1 - x_2| < q_n$$

It follows that since $p > \gamma^2$ we can use the same argument. That completes the proof of Lemma 5.2. ∎

Let us define $m'(m, L_0, L_1, L_2) = m - \bar{m}(m, L_0, L_1, L_2)$. It can be easily seen from the definition of the sequence of scales L_i and (2.1) that for any $L_1 < \infty$ and $m > 0$ we can find J_0, m_0 such that for $J < J_0$ the scale L_1 is m_0-simple and

$$\sum_{\substack{i: i = j_1(n) - 1 \text{ for some } n \\ \text{or } q_{n(i)+1} < q^s_{n(i)}}} m'(m_i, L_i, L_i, L_{i+1})$$
$$+ \sum_{n: q_{n+1} > q^s_n} (m'(m_{j_1(n)}, L_{j_0(m+1)-3}, L_{j_0(m+1)-2}, L_{j_0(m+1)-1})$$
$$+ m'(m_{j_0}(m+1) - 1, L_{j_0(m+1)-3}, L_{j_0(m+1)-1}, L_{j_0(m+1)})) < m_0 - m$$

Lemma 4.3 now follows by induction from Lemmas 5.1 and 5.2. ∎

Proof of Lemma 5.1. We will refer to $n(i)$ as n unless otherwise noted. The cases (i) and (ii) are "almost diophantine" and so is the argument.

Proof of (i). Suppose there exist three (L_i, m_i)-singular points $x_1, x_2, x_3 \in \Lambda_{L_{i+1}}$. Since L_i is m_i-simple, we can find L_i-resonant points $\bar{x}_1, \bar{x}_2, \bar{x}_3$, with $\bar{x}_i \in \Lambda_{L_i}(x_i)$, $i = 1, 2, 3$. We conclude that

$$d(T^{\bar{x}_i}\theta, T^{\bar{x}_j}\theta) < 2L_i^{-p}, \qquad i, j = 1, 2, 3 \tag{5.1}$$

and for at least one pair (i, j) the distance $|\bar{x}_i - \bar{x}_j| < L_{i+1} < q_{n+1}$. Without loss of generality we assume that $|\bar{x}_1 - \bar{x}_2| < q_{n+1}$; thus we get by (4.1), (4.2) that

$$d(T^{\bar{x}_1}\theta, T^{\bar{x}_2}\theta) > \frac{1}{2q_{n+1}} > \frac{1}{2q^s_n}$$

If $|\bar{x}_1 - \bar{x}_2| > q_n$, we have

$$d(T^{\bar{x}_1}\theta, T_{\bar{x}_2}\theta) > \frac{1}{2|x_1 - x_2|^s}$$

Thus $|\bar{x}_1 - \bar{x}_2|^s > \frac{1}{4} L_i^p$, which, since $p > \gamma s$, is in contradiction with $|x_1 - x_2| < L_{i+1}$. If $0 < |\bar{x}_1 - \bar{x}_2| < q_n$, we have

$$d(T^{\bar{x}_1}\theta, T^{\bar{x}_2}\theta) > \frac{1}{2q_n}$$

which, together with (5.1), is in contradiction with $L_i > q_n$. If we now suppose that the box $\Lambda_{L_{i+1}}(x)$ is nonresonant, then we can apply Theorem 2.3e to get that it is m_{i+1}-regular. Thus the scale L_{i+1} is m_{i+1}-simple.

Proof of (ii). Analogous arguments show that for any two L_i-resonant points $\bar{x}_1$ and $\bar{x}_2$ such that $|\bar{x}_1 - \bar{x}_2| < L_{i+1}$ we have (5.1). If $0 < |\bar{x}_1 - \bar{x}_2| < q_{n+1}$, we may use the same argument as above. If $|\bar{x}_1 - \bar{x}_2| > q_{n+1}$, we use that

$$|\bar{x}_1 - \bar{x}_2| < L_{i+1} < q_{n(i+1)+1} < q^s_{n(i+1)} < L^s_{i+1} \leqslant L_i^{\gamma s}$$

Thus

$$d(T^{\bar{x}_1}\theta, T^{\bar{x}_2}\theta) > \frac{1}{2q_{n(i+1)+1}} > \frac{1}{2L_i^{\gamma s}}$$

This is in contradiction with (5.1) and the proof can be completed as above.

In the cases (iii) and (iv) we cannot apply Theorem 2.3, so we will use the following:

Sublemma 5.3. Let $q_{n(i)+1} > q^s_{n(i)}$. Suppose there exists $\bar{x} \in \Lambda_{L_{i+1}}(y)$ such that $\Lambda_{L_{i+1}}(y) \backslash \Lambda_{2L_i}(\bar{x})$ is (m_{i_0}, L_{i_0})-regular for some $i_0 \leqslant i$. Suppose also that for some $\delta < \delta' < 1 - \delta$, $\delta < \kappa < 1 - \delta'$ we have

$$\sum_{x \in \Lambda_{L_i q^{r\delta'}_{n(i)}}(x)} \ln h(x) - JL_i q^{r\delta'}_{n(i)} > -L_{i+1} q^{-r\kappa}_{n(i)} \tag{5.2}$$

Fix $r(1-\delta') < \tau < r(1-\delta)$. Then there exists $\bar{l} = \bar{l}(m_i, L_i, r, \delta, \delta', \kappa, \tau)$ such that if $L_i > \bar{l}$, we have

$$G_{B_{L_{i+1}}(y,0)}((y, 0), Y) \leqslant \exp[-M \exp(1/2 L_i q^{r-\tau}_{n(i)})]$$

for all $Y \in \partial_H B_{L_{i+1}}((y, 0))$ with $M \geqslant m - \exp(-\frac{1}{4} L_i q^{r-\tau}_{n(i)})$.

Proof of Sublemma 5.3. We will follow the proof of the analogous statement in ref. 4. It follows from our construction that under the conditions of the lemma we have $L_{i+1} = L_i q^r_{n(i)}$ and $T(L_{i+1}) = \exp(L_{i+1} q_{n(i)}^{-r\delta})$.

Take

$$Y = (z, e^{L_{i+1} q_{n(i)}^{-r\delta}}), \qquad z \in \Lambda_{L_{i+1}}(y)$$

The case

$$Y = (z, -e^{L_{i+1} q_{n(i)}^{-r\delta}})$$

can be treated in the same way. We set

$$S_j = B_{L_{i+1}, 1/2 \exp(L_{i+1} q_{n(i)}^{-\tau})}(y, (j-1/2)\, e^{L_{i+1} q_{n(i)}^{-\tau}}), \qquad j = 1, 2,..., [e^{L_{i+1}(q_{n(i)}^{-r\delta} - q_{n(i)}^{-\tau})}]$$

Denote

$$\Lambda_{L_i q_{n(i)}^{r\delta'}}(\bar{x})$$

by $\tilde{\Lambda}$. Set

$$H_s = \tilde{\Lambda} \times [s - \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}, s + \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}]$$

For each s we introduce the event D_s given by

$$D_s = \{\text{there are no bridges in } H_s \text{ and for each } x \in \tilde{\Lambda} \text{ the line segment } \{x\} \times [s - \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}, s + \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}] \text{ has at least one cut}\}$$

For each configuration in D_s there is no connection between

$$\tilde{\Lambda} \times \{s - \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}\} \qquad \text{and} \qquad \tilde{\Lambda} \times \{s + \tfrac{1}{2} e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}\}$$

inside H_s. We have

$$\begin{aligned} Q(D_s) &= \exp[-J|\tilde{\Lambda}|\, e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}] \prod_{x \in \tilde{\Lambda}} \{1 - \exp[-h(x)\, e^{-q_{n(i)}^{r(1-\delta'-\kappa)}}]\} \\ &\geqslant \exp[-L_i q_{n(i)}^{r\delta'}(J e^{-q_{n(i)}^{r(1-\delta'-\kappa)}} + q_{n(i)}^{r(1-\delta'-\kappa)} - \ln 2) + \sum_{x \in \tilde{\Lambda}} \ln h(x)] \\ &\geqslant \exp[-2L_{i+1} q_{n(i)}^{-r\kappa}] \end{aligned}$$

by (5.2).

Let us denote

$$B_\Lambda = \Lambda \times [-e^{L_{i+1} q_{n(i)}^{-r\delta}}, e^{L_{i+1} q_{n(i)}^{-r\delta}}]$$

for any $\Lambda \subset \mathbf{Z}$. Let $\hat{\Lambda} = \Lambda_{2L_i}(\bar{x})$ and let F_j be the event that there is no connection inside $S_j \backslash B_{\hat{\Lambda}}$ from the exterior boundary of $B_{\hat{\Lambda}}$ to $B_{\Lambda_{L_{i+1}} \backslash \hat{\Lambda}}$. Since $S_j \backslash B_{\hat{\Lambda}}$ is entirely inside an (m_{i_0}, L_{i_0})-regular region we have that

$$\begin{aligned} Q(F_j^c) &\leqslant 2\exp[2L_{i+1} q_{n(i)}^{-\tau}] \exp\left[-m_{i_0} L_{i_0}\left(\frac{L_i q_{n(i)}^{r\delta'} - 2(L_i+2)}{L_{i_0}+1} - 1\right)\right] \\ &\leqslant 2\exp[2L_i q_{n(i)}^{r-\tau} - m_i L_i (q_{n(i)}^{r\delta'} - 2)] \\ &\leqslant \exp[-cL_i q_{n(i)}^{r\delta'}] \end{aligned}$$

for some $c = c(r, \tau, \delta', m_i)$ and L_i sufficiently large, since $\tau > r(1-\delta')$.

We now define

$$A_j = F_j \cap D_{(j-1/2)\exp(L_{i+1} q_{n(i)}^{-\tau})}, \qquad j = 1, \ldots, [e^{L_{i+1}(q_n^{-r\delta} - q_n^{-r})}]$$

Both F_j and

$$D_{(j-1/2)\exp(L_{i+1} q_{n(i)}^{-r})}$$

are local negative events; thus the Harris–FKG inequality implies

$$\begin{aligned} Q(A_j) &\geqslant Q(F_j)\, Q(D_{(j-1/2)\exp(L_{i+1} q_{n(i)}^{-\tau})}) \\ &\geqslant (1 - e^{cL_i q_{n(i)}^{r\delta'}})\, e^{-2L_i q_{n(i)}^{r(1-\kappa)}} \geqslant e^{-3L_i q_{n(i)}^{r(1-\kappa)}} \end{aligned}$$

Let

$$A = \bigcup A_j, \qquad j = 1, \ldots, [e^{L_{i+1}(q_{n(i)}^{-r\delta} - q_{n(i)}^{-r})}]$$

All A_j are independent identically distributed events and we get

$$\begin{aligned} Q(A^C) &= \prod (1 - Q(A_j)) \\ &= (1 - Q(A_j))^{\exp[L_{i+1}(q_{n(i)}^{-r\delta} - q_{n(i)}^{-\tau})]} \\ &\leqslant \{1 - \exp[-3L_i q_{n(i)}^{r(1-\kappa)}]\}^{\exp[L_i(q_{n(i)}^{r(1-\delta)} - q_{n(i)}^{r-\tau}) - 1]} \\ &\leqslant 3\exp\{-\exp[-3L_i q_{n(i)}^{r(1-\kappa)} + L_i q_{n(i)}^{r(1-\delta)} - L_i q_{n(i)}^{r-\tau}]\} \\ &\leqslant \exp\{-\exp[1/2 q_{n(i)}^{r(1-\delta)} L_i]\} \end{aligned}$$

since $\kappa > \delta$ and $\tau > r(1-\delta') > r\delta$, for L_i sufficiently large. We have

$$\{0 \leftrightarrow_{B_{L_{i+1}}(y,0)} Y\} \cap A \subset C$$

where C is the event that there exists a connection of vertical length $\geqslant \exp(L_{i+1} q_{n(i)}^{-\tau})$ inside an (m_{i_0}, L_{i_0})-regular region $B_{\Lambda_{L_{i+1}}(y)\backslash\hat{\Lambda}}$. Evidently

$$\begin{aligned} Q(C) &\leqslant |B_{\Lambda_{L_{i+1}}(y)\backslash\hat{\Lambda}}|^2 \exp(-m_{i_0} L_{i_0}\{[\exp(L_{i+1} q_{n(i)}^{-\tau})][T(L_{i_0})]^{-1} - 1\}) \\ &\leqslant [L_{i+1} \exp(L_{i+1} q_{n(i)}^{-r\delta})]^2 \exp[-m_{i_0} L_{i_0} \exp(L_i q_{n(i)}^{r-\tau} - L_{i_0} q_{n(i_0-1)}^{-r\delta})] \\ &\leqslant \exp[-M' \exp(L_i q_{n(i)}^{r-\tau})] \end{aligned}$$

Thus

$$G_{B_{L_{i+1}}(y,0)}((y, 0), Y) \leqslant \exp[-M \exp(1/2 L_i q_{n(i)}^{r-\tau})]$$

with

$$M \geqslant m - \exp(-1/4 L_i q_{n(i)}^{r-\tau}) \quad \blacksquare$$

Now we start the proof of case (iii) of Lemma 5.1. For any x_1, x_2 such that $|x_1 - x_2| < L_{i+1} < q_{n(i+1)+1}$ we get

$$d(T^{x_1}\theta, T^{x_2}\theta) > \frac{1}{2q_{n(i+1)+1}} > \frac{1}{2q_{n(i+1)}^s}$$

On the other hand, if we suppose that x_1, x_2 are L_i-resonant, then (5.1) is satisfied and

$$d(T^{x_1}\theta, T^{x_2}\theta) < \frac{4}{L_i^p} = \frac{4q_{n(i)}^{rp}}{L_{i+1}^p} \leqslant \frac{4q_{n(i)+1}^{rp/s}}{q_{n(i+1)}^p} \leqslant \frac{4}{q_{n(i+1)}^{p(1-r/s)}}$$

We get that since $p(1 - r/s) > s$, then if L_i is large enough, $|x_1 - x_2| > L_{i+1}$. Now we only need to prove (5.2) for nonresonant $\Lambda_{L_{i+1}}(y)$ and some appropriate values of δ', κ in order to be able to use Sublemma 5.3, which will allow us to complete the proof in the same way as before. Let us denote $d(T^x\theta, 0)$ by $d(x)$. Our condition on the function $f(\theta)$ implies that $\ln h(x) > -(d(x))^{-\eta} + c$; thus, in order to apply Sublemma 5.3, we are to estimate

$$\sum_{x \in \Lambda_{L_i q_{n(i)}^{r\delta'}}(\bar{x})} d(x)^{-\eta}$$

for some $\delta < \delta' < 1 - \delta$. Suppose $L_{i+1}(y)$ is nonresonant; then $d(x) \geqslant L_{i+1}^{-\mu/\eta}$ for $x \in \Lambda_{L_{i+1}}(y)$ and (4.1), (4.2) give us the following estimate:

$$\sum_{x \in \Lambda_{L_i q_{n(i)}^{r\delta'}}(\bar{x})} d(x)^{-\eta} \leqslant 4 \sum_{k=0}^{1/2L_i q_{n(i)}^{r\delta'} q_{n(i)+1}^{-1}} \left(L_{i+1}^{-\mu/\eta} + \frac{k}{2q_{n(i+1)+1}} \right)^{-\eta}$$
$$+ 4 \sum_{k=1}^{1/2L_i q_{n(i)}^{r\delta'-1}} \left(L_{i+1}^{-\mu/\eta} + \frac{k}{2q_{n(i)+1}} \right)^{-\eta}$$
$$+ 4 \sum_{k=1}^{1/2L_i q_{n(i)}^{r\delta'}} \left(L_{i+1}^{-\mu/\eta} + \frac{k}{2q_{n(i)}} \right)^{-\eta} = \Sigma^1 + \Sigma^2 + \Sigma^3$$

We will estimate Σ^1, Σ^2, and Σ^3 separately.

1. We have

$$\Sigma^1 \leqslant 4L_{i+1}^{\mu} \sum_{k=0}^{1/2L_i q_{n(i)}^{r\delta'} q_{n(i)+1}^{-1}} \left(1 + \frac{kL_{i+1}^{\mu/\eta}}{2q_{n(i+1)}^{s}} \right)^{-\eta}$$
$$\leqslant 4L_{i+1}^{\mu} \sum_{k=0}^{1/2L_i q_{n(i)}^{r\delta'} q_{n(i)+1}^{-1}} \left(1 + \frac{k}{2} L_{i+1}^{\mu/\eta - s} \right)^{-\eta}$$
$$\leqslant 4L_{i+1}^{\mu} \left[1 + \int_0^{1/2L_i q_{n(i)}^{r\delta'} q_{n(i)+1}^{-1}} (1 + 1/2L_{i+1}^{\mu/\eta - s} x)^{-\eta}\, dx \right]$$
$$\leqslant 4L_{i+1}^{\mu} \left\{ 1 + \frac{2}{L_{i+1}^{\mu/\eta - s}(1-\eta)} \left[(1 + 1/4 q_{n(i)}^{r\delta'} L_{i+1}^{\mu/\eta - s})^{1-\eta} - 1 \right] \right\}$$
$$\leqslant 4L_{i+1}^{\mu} q_{n(i)}^{r\delta'} < 4L_{i+1} q_{n(i)}^{-s(1-\mu) + r\delta'}$$

Here we used that $L_{i+1} > q_{n(i)+1} > q_{n(i)}^{s}$.

2. In the analogous way we get

$$\Sigma^2 \leqslant 4L_{i+1}^{\mu} \sum_{k=1}^{1/2L_i q_{n(i)}^{r\delta'-1}} \left(1 + \frac{kL_{i+1}^{\mu/\eta}}{2q_{n(i)+1}} \right)^{-\eta}$$
$$\leqslant 4L_{i+1}^{\mu} \sum_{k=1}^{1/2L_i q_{n(i)}^{r\delta'-1}} (1 + k/2L_{i+1}^{\mu/\eta - 1})^{-\eta}$$
$$\leqslant 4L_{i+1}^{\mu} \int_0^{1/2L_i q_{n(i)}^{r\delta'-1}} \frac{dx}{(1 + (L_{i+1}^{\mu/\eta - 1}/2)x)^{\eta}}$$
$$\leqslant \frac{8L_{i+1}^{\mu}}{L_{i+1}^{\mu/\eta - 1}(1-\eta)} \left[\left(1 + \frac{1}{4} q_{n(i)}^{r\delta'-1} L_i L_{i+1}^{\mu/\eta - 1} \right)^{1-\eta} - 1 \right]$$
$$\leqslant \frac{8}{1-\eta} L_{i+1}^{\mu(1-1/\eta)+1} q_{n(i)}^{(r\delta'-1)(1-\eta)} L_{i+1}^{\mu/\eta - \mu} q_{n(i)}^{-r(1-\eta)}$$
$$\leqslant 8(1-\eta)^{-1} L_{i+1} q_{n(i)}^{-(1-\eta)(1+r(1-\delta'))}$$

3. To estimate Σ^3, we write

$$\begin{aligned}\Sigma^3 &\leqslant 4 \sum_{k=1}^{1/2L_i q_{n(i)}^{r\delta'}} \left(\frac{k}{2q_{n(i)}}\right)^{-\eta} \\ &\leqslant \frac{2^{2+\eta} q_{n(i)}^{\eta}}{1-\eta} (1/2L_i q_{n(i)}^{r\delta'})^{1-\eta} \\ &\leqslant \frac{8}{1-\eta} L_{i+1}^{1-\eta} q_{n(i)}^{r(\delta'-1)(1-\eta)+\eta} \\ &< \frac{8}{1-\eta} L_{i+1} q_{n(i)}^{-\eta(s-1)-r(1-\delta')(1-\eta)}\end{aligned}$$

Since

$$\sum_{x \in \Lambda_{L_i q_{n(i)}^{r\delta'}}(\bar{x})} \ln h(x) - J L_i q_{n(i)}^{r\delta'} > \Sigma^1 + \Sigma^2 + \Sigma^3 - (J-c) L_i q_{n(i)}^{r\delta'}$$

we get that if we take

$$\delta < \delta' < \min\left(\frac{s(1-\mu)}{r}, 1\right) - \delta$$

$$\delta < \kappa < \min\left(1-\delta', \frac{s(1-\mu)}{r} - \delta', \frac{\eta(s-1)}{r} + (1-\delta')(1-\eta), \frac{(1-\eta)}{r} + (1-\eta)(1-\delta')\right) \tag{5.3}$$

then the condition (5.2) of Sublemma 5.3 will be satisfied.

The value of κ satisfying (5.3) can be found $\delta' < 1 - \delta$ implies

$$1 - \delta' > \delta, \qquad \eta(s-1)r^{-1} + (1-\delta')(1-\eta) > \delta$$

Furthermore,

$$\frac{1-\eta}{r} + (1-\eta)(1-\delta') > \delta, \qquad \delta' < s(1-\mu)r^{-1} - \delta$$

implies $s(1-\mu)r^{-1} - \delta' > \delta$.

We can now use Sublemma 5.3 to prove the desired decay of the two-point function and thus to conclude that L_{i+1} is m_{i+1}-simple, which finishes the proof of statement (iii).

Before we start the proof of the last statement (iv) of the lemma, we will need the following:

Proposition 5.4. For $q_i < k < q_{i+1}/2$ we have

$$d(T^k\theta, \theta) \geqslant \left(\left[\frac{k}{q_i}\right] - 1\right) \Delta_i$$

Proof. Let us represent k as $k = b_i q_i + \cdots + b_1 q_1 + b_0$, where

$$b_j = \left[\frac{k - b_i q_i - \cdots - b_{j+1} q_{j+1}}{q_j}\right] \quad \text{for} \quad 0 \leqslant j < i, \quad \text{and} \quad b_i = \left[\frac{k}{q_i}\right]$$

Evidently $0 \leqslant b_j \leqslant k_{j+1}$. We have

$$d(T^k\theta, \theta) = d(T^k 0, 0) = \left| \sum_{j=0}^{i} b_j d(T^{q_j} 0, 0)(-1)^j \right|$$

Here we used that $\operatorname{sign}(\omega q_j - p_j) = -\operatorname{sign}(\omega q_{j+1} - p_{j+1})$. Denote $\omega q_j - p_j$ by a_j. Recall that $d(T^{q_j} 0, 0) = |\omega q_j - p_j| = \Delta_j$. Since $q_j = k_j q_{j-1} + q_{j-2}$, $p_j = k_j p_{j-1} + p_{j-2}$, we have the same relations for the sequence a_j: $a_j = k_j a_{j-1} + a_{j-2}$, which implies $\Delta_j = |a_j| = \Delta_{j-2} - k_j \Delta_{j-1}$. Thus

$$\Delta_j = \Delta_{j-2l} - \sum_{r=1}^{l} k_{j-2l+2r} \Delta_{j-2l+2r-1} \qquad \text{for any} \quad l < j/2$$

Let $j_0 = \min\{j \geqslant 0 \colon b_j \neq 0\}$.

1. If $(i - j_0)/2 \in N$, then

$$\begin{aligned} d(T^k\theta, \theta) &\geqslant b_{j_0} \Delta_{j_0} - \sum_{r=1}^{(i-j_0)/2} b_{j_0+2r} \Delta_{j_0+2r-1} + b_i |a_i| \\ &\geqslant b_{j_0} \Delta j_0 - \sum_{r=1}^{(j-j_0)/2} k_{j_0+2r} \Delta_{j_0+2r-1} + b_i \Delta_i \\ &\geqslant (b_{j_0} - 1) \Delta_{j_0} + b_i \Delta_i \geqslant b_i \Delta_i \end{aligned}$$

Since $b_{i-1} q_{i-1} + \cdots + b_1 q_1 + b_0 < q_i$, we have

$$d(T^k\theta, \theta) \geqslant \left[\frac{k}{q_i}\right] \Delta_i$$

2. If $(i - j_0 + 1)/2 \in \mathbf{N}$, then in an analogous way we get

$$\begin{aligned} d(T^k\theta, \theta) &\geqslant b_{j_0} \Delta_{j_0} - \sum_{r=1}^{(i-j_0-1)/2} k_{j_0+2r} \Delta_{j_0+2r-1} - b_i \Delta_i \\ &\geqslant (b_{j_0} - 1) \Delta_{j_0} + \Delta_{i-1} - b_i \Delta_i \\ &\geqslant \Delta_{i-1} - b_i \Delta_i \\ &\geqslant (k_{i+1} - b_i) \Delta_i \end{aligned}$$

Since $k < q_{i+1}/2$ and $b_i = [k/q_i]$, we have

$$b_i < \frac{q_{i+1}}{2q_i} < \frac{1}{2}\left(\left[\frac{q_{i+1}}{q_i}\right] + 1\right) \leqslant \frac{1}{2}(k_{i+1} + 1)$$

Thus

$$d(T^k\theta, \theta) \geqslant \frac{1}{2}(k_{i+1} - 1)\Delta_i \geqslant (b_i - 1)\Delta_i \geqslant \left(\left[\frac{k}{q_i}\right] - 1\right)\Delta_i \quad \blacksquare$$

We now return to the proof of (iv) of Lemma 5.1. There will be no further interruptions. We will refer to $n(i+1)$ as n and $j_0(n+1)$ as j_0.

Since $L_{j_0} < q_{n+1}/2$, applying Proposition 5.4, we get that for all $x \in \Lambda_{L_{j_0}}(0) \backslash \Lambda_{q_n}(0)$,

$$d(T^x\theta, \theta) \geqslant \left(\left[\frac{x}{q_n}\right] - 1\right)\Delta_n$$

From $L_{j_0+1} = L_{j_0} q_n^r > q_{n+1}/2$, we have the following estimates: $L_{j_0} > q_{n+1}/2q_n^r$, $L_{j_0-2} > q_{n+1}/2q_n^{3r}$. Thus for large n and any $L_{j_0-2} < |x| < L_{j_0}$ the quantity

$$\left(\left[\frac{x}{q_n}\right] - 1\right)\Delta_n \geqslant \left(\frac{q_{n+1}}{2q_n^{3r+1}} - 1\right)\frac{1}{2q_{n+1}} \geqslant \frac{1}{8q_n^{3r+1}} \tag{5.4}$$

In the same way as before we get (5.1) for any L_i-resonant points x_1 and x_2.

We now consider two cases.

1. $n(i+1) = n(i)$. Then we have

$$\frac{1}{L_i^p} < \frac{1}{q_{n(i)}^p} < \frac{1}{8q_{n(i+1)}^{3r+1}} \tag{5.5}$$

Suppose $x_1, x_2 \in \Lambda_{L_{j_0-1}+L_{j_0-2}}(y)$. Then (5.1), (5.4), and (5.5) imply that $|x_1 - x_2| < L_{j_0-2}$. Thus for any y there exist $\bar{x} \in \Lambda_{L_{j_0-1}}(y)$ such that $\Lambda_{L_{j_0-1}}(y) \backslash \Lambda_{(3/2)L_{j_0-2}}(\bar{x})$ is (m_i, L_i)-regular. We now need to prove the condition (5.2) for nonresonant box $\Lambda_{L_{j_0-1}}(y)$ in order to apply Sublemma 5.3. In the same way as for the case (iii), we get that for every x in a nonresonant box $\Lambda_{L_{j_0-1}}(y)$ we have $d(x) \geqslant L_{j_0-1}^{-\mu/\eta}$ and we want to estimate

$$\sum_{x \in \Lambda_{L_{j_0} - 2q_{n(i+1)}^{r\delta'}}(\bar{x})} d(x)^{-\eta}$$

for some $\delta < \delta' < 1 - \delta$.

Using again (4.1), (4.2), we get

$$\sum_{x \in \Lambda_{L_{j_0}-2q^{r\delta'}_{n(i+1)}}(\bar{x})} d(x)^{-\eta} \leqslant L^{\mu}_{j_0-1} + 4 \sum_{k=1}^{1/2 L_{j_0}-2q^{r\delta'}_{n(i+1)}} \left(\frac{k}{q_{n(i+1)+1}}\right)^{-\eta}$$

$$\leqslant L^{\mu}_{j_0-1} + \frac{4}{(1-\eta)2^{1-\eta}} (L_{j_0}-2q^{r\delta'}_{n(i+1)})^{1-\eta} q^{\eta}_{n(i+1)+1} \tag{5.6}$$

Since

$$L_{j_0}-2q^{3r}_{n(i+1)} = L_{j_0+1} > \frac{q_{n(i+1)+1}}{2}$$

we can estimate the right-hand side of (5.6) as

$$L^{\mu}_{j_0-1} + \frac{8}{1-\eta} L_{j_0-1} q^{r(\delta'(1-\eta)+3\eta-1)}_{n(i+1)}$$

If we now pick $\delta < \delta' < (1-3\eta-\delta)/(1-\eta)$, which is possible since $\delta < (1-3\eta)/(1-\eta)$, and $\delta < \kappa < 1-3\eta-\delta'(1-\eta)$, we will fulfill the condition (5.2) and thus prove that L_{j_0-1} is m_{j_0-1}-simple. The same argument works for L_{j_0}.

2. Let us now turn to the case $n(i+1) > n(i)$: If $L_{i+1} = L^{\gamma}_i$, we have

$$\frac{1}{L^p_i} = \frac{1}{L^{p/\gamma}_{i+1}} < \frac{1}{q^{p/\gamma}_{n(i+1)}} < \frac{1}{8q^{3r+1}_{n(i+1)}}$$

since $p > \gamma(3r+1)$ and L_i is large enough. We conclude in the same way as above that $|x_1 - x_2| < L_{j_0-2}$.

If $L_{i+1} = L_i q^r_{n(i)}$, we have $L_{i+1} > q_{n(i+1)}$, $n(i+1) \geqslant n(i)+1$; thus

$$\frac{1}{L^p_i} < \frac{q^{rp}_{n(i)}}{q^p_{n(i+1)}} < \frac{q^{rp/s}_{n(i+1)}}{q^p_{n(i+1)}} < \frac{1}{q^{p(1-r/s)}_{n(i+1)}} < \frac{1}{8q^{3r+1}_{n(i+1)}}$$

since

$$p > \frac{(3r+1)s}{s-r}$$

Thus $|x_1 - x_2| < L_{j_0-2}$. After this the rest of the proof is the same as in case 1. ▮

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